In the DC analysis part of VE215, you will encounter capacitors and inductors, whose voltage-current relationship does not satisfy Ohm’s law. Their relationships, characterized by differential equations, cannot be solved with methods with which we use to solve circuits with resistors only. Solving first-order circuits (RC circuits and RL circuits) may not be very difficult with the variable separation method, but second-order circuits (RLC circuits) are much more complex, as they are characterized by second-order inhomogeneous linear ODEs. Based on王高雄 等，《常微分方程（第三版）》，高等教育出版社, this article aims to provide instructions on how to solve high-order inhomogeneous linear ODEs, which will cover some of the most difficult calculation parts of VE215 with a more generalized and mathematical approach.

4.1. The General Theories of Linear ODEs.

We look at an nth-order linear ODE

, (4.1),

where and are both continuous functions over .

If , equation (4.1) turns into

, (4.2).

We call it an nth-order homogeneous linear ODE, while equation (4.1) is inhomogeneous.

Theorem 1. The Existence and Uniqueness of the Solution.

If and are both continuous functions over , for and , there exists a unique solution to equation (4.1), which is defined over , and satisfies the initial condition .

Theorem 2. The Superposition Principle.

If are k solutions to equation (4.2), their linear combination is also a solution to equation (4.2), where are arbitrary constants.

Theorem 5.

There must exist n linear independent solutions to equation (4.2).

Theorem 6. The Structure of the General Solution.

If are n linear independent solutions to equation (4.2), the general solution to equation (4.2) can be characterized by

, (4.11),

where are arbitrary constants.

Property 1.

If is a solution to equation (4.1), and is a solution to equation (4.2), is also a solution to equation (4.1).

Property 2.

The difference between any two solutions to equation (4.1) must be a solution to equation (4.2).

Theorem 7.

Suppose that is the fundamental solution set of equation (4.2), and is a solution to equation (4.1), the general solution to equation (4.1) can be expressed as

, (4.14),

where are arbitrary constants. The general solution (4.14) includes all solutions to equation (4.1).

4.2. The Methods of Solving Linear ODEs with Constant Coefficients.

Theorem 8.

If all coefficients in equation (4.2) are real-value functions, and is a complex-value solution, , , and are all solutions to equation (4.2).

Theorem 9.

If has a complex-value solution , where , , and are all real-value functions, is the solution to and is the solution to .

Suppose that all the coefficients in a homogeneous linear ODE are constants, which means

, (4.19),

where are constants. We call this an nth-order homogeneous linear ODE with constant coefficients. To find the general solution of equation (4.19), we only need to solve for its fundamental solution set. Next we will discuss the coefficient comparison method, or the eigen root method.

, (4.21),

is denoted as the eigen equation of equation (4.19). Its roots are called eigen roots. Next we will discuss situations where the multiplicity of the eigen roots are different respectively.

In the first case, all eigen roots are of multiplicity 1.

Suppose that are n different roots of the eigen equation (4.21), equation (4.19) has the following n roots:

, (4.22).

We point out that these n roots are linear independent over , and they form the fundamental solution set.

If are all real numbers, the solutions (4.22) are n linear independent real-value solutions to equation (4.19), so its general solution can be expressed as , where are arbitrary constants.

If the eigen equation has complex roots, the complex roots will appear in conjugate pairs because the coefficients of the equation are real constants. Suppose that is an eigen root, is also an eigen root. Correspondingly, equation (4.19) has two complex-value solutions, and . According to Theorem 8, their real parts and imaginary parts are also solutions to the equation. Thus, for a conjugate pair root , we can solve equation (4.19) for two real-value solutions, and .

In the second case, there exist some eigen roots whose multiplicities are larger than 1.

If is an eigen value of multiplicity k, is also an eigen value of multiplicity k. Therefore, we have 2k real-value solutions: .

Example 1.

Find the general solution to .

The roots of the eigen function are . There are two real roots and two complex roots without repetitions, so the general solution is , where are arbitrary constants.

Example 2.

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The eigen equation has roots . Therefore, the general solution is , where are arbitrary constants.

Example 3.

Find the general solution to .

The eigen function is , or . This means that , so the general solution is , where are arbitrary constants.

Example 4.

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The eigen function is , or . This means that . Therefore, the equation has 4 real-value solutions: . The general solution is , where are arbitrary constants.

Now we will look at the method of solving an inhomogeneous linear ODE with constant coefficients,

, (4.32),

where are constants, and is continuous.

We will only cover the coefficient comparison method of solving this equation when satisfies some special forms.

In the first case, , where and are real constants, and equation (4.32) has a special solution of the form

, (4.33),

where k is the multiplicity of the root λ of the eigen equation (let if λ is not an eigen root), and are the constants to be determined.

Example 7.

Find the general solution to .

First, we should find the general solution to the corresponding homogeneous linear ODE . Here, the eigen equation has two roots . Consequently, the general solution is , where are arbitrary constants. Then, we will find a particular solution to the inhomogeneous linear ODE. Here, we have and . Since is not an eigen root, we assume that the particular solution is of the form , where are the constants to be determined. To determine , we replace the original x with the particular solution, and we get . Compare the coefficients, and we have . Therefore, we have , so . Finally, the general solution of the original equation is .

Example 8.

Find the general solution to .

From Example 7 we know that the general solution to the homogeneous linear ODE is , where are arbitrary constants. Now we will find a particular solution to the equation, with . Since is a root of multiplicity 1, we have a particular solution of the form . Replace the original x, and we get , which means that . Therefore, , and the general solution to the original equation is given by .

Example 9.

Find the general solution to .

The eigen equation has a root of multiplicity 3, , so the homogeneous linear ODE’s general solution is given by . The inhomogeneous linear ODE has a particular solution of the form . Replace the original x, and we get . Compare the coefficients, and we get , with . Consequently, the general solution the equation is , where are arbitrary constants.

In the second case, , where are constants, and are polynomials with real coefficients, one of whose degree is m, and the other of whose degree is no larger than m. In this case, equation (4.32) has a particular solution of the form

, (4.38),

where k is the multiplicity of the root of the eigen equation, and are both polynomials with real coefficients whose degrees are not larger than m, to be determined by the coefficient comparison method.

Note that should both be supposed to be complete polynomials of degree m in the actual calculations.

Example 10.

Find the general solution to .

The eigen equation has a root of multiplicity 2, . Consequently, the general solution to the homogeneous linear ODE is given by , where are arbitrary constants. Now we will look for a particular solution to the inhomogeneous linear ODE. Since ±2i is not an eigen root, the particular solution has a form of . Replace the original x, and we have . Compare the coefficients, and we have . With , the general solution to the equation is .

Note that a special subcase of the second case or can be solved by a simpler method, the so-called complex number method.

Example 11.

Solve Example 10 with the complex number method.

From Example 10 we know that the general solution to the homogeneous linear ODE is . To find a particular solution to the inhomogeneous linear ODE, we need to find a particular solution to first. This question falls in the first case. Since 2i is not an eigen root, the particular solution can be of the form . Replace the original x, and we get . Therefore, . According to Theorem 9, is a particular solution to the original equation. Consequently, the general solution is , and we have the same result as Example 10.

This article provides some theoretical knowledge about general linear ODEs and some examples on solving high-order inhomogeneous linear ODEs with constant coefficients. For the DC analysis part of VE215 this should be sufficient. The proofs of theorems and properties are not provided, because this course is not about mathematics despite a lot of calculations. More theories about ODEs are given in VV256, including the Wronskian determinant and Laplace transformation. Interested readers are encouraged to look for more materials by themselves.